

Output Regulation and Consensus of a Class of Multi-agent Systems under Switching Communication Topologies

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Abstract. This paper presents the design of a distributed control law for the output regulation and output consensus of a set of N agents. In this approach, each agent dynamics is represented by a switched linear system. The representation of the agents is neither constrained to be the same nor to have the same state dimension, and communication among agents is considered to be switching. It is also considered that some agents get the reference to be followed from the output of a virtual agent, and every agent gets the output information of its neighbors. Using this information, every agent computes the exosystem state to solve its individual regulation problem. The approach herein proposed employs a local switched stabilizing feedback for each agent based on a common Lyapunov function. A numerical example is provided in order to illustrate the proposed control law.

Keywords. Distributed control, multi-agents, switched linear systems, regulation.

1 Introduction

In recent years, consensus in multi-agent systems (*MAS*) has attracted considerable attention in Computer Science and Control Theory since it allows to solve complex problems such as formation control with trajectory tracking.

Some previous results related to consensus problems in *MAS* are introduced in [14] and [4], where an overview of methods for convergence and analysis of consensus algorithms are presented. The development of consensus algorithms for linear and nonlinear systems is presented in

[15] and [12], and a Lyapunov function is proposed to study the convergence of the consensus algorithms. In [6] and [20], necessary and sufficient conditions for an appropriate decentralized linear stabilizing feedback are established.

In [16], the problem of *flocking* with obstacles is addressed, where flocking is defined as achieving both structural and navigational stability. The systems considered are restricted to have integrator dynamics, and stability results are not presented.

A feedback control strategy that achieves convergence of a *MAS*, for single-integrator dynamics, with a desired formation and avoiding collisions is proposed in [1]. A connection between formation infeasibility and a sort of flocking is established. Consensus of multiple autonomous vehicles is addressed in [7], by using virtual leaders and artificial potential fields among neighboring vehicles. Results are also restricted to vehicles with integrator dynamics. A decentralized dynamic controller dealing with the problem of cooperation among a collection of vehicles is presented in [3] and [9].

The problems of consensus (synchronization), model-reference, and regulation for a network of identical multi-input, multi-output linear *MAS* are considered in [22]. That work proposes a distributed protocol to solve such problems for network Laplacian topologies and asymmetric topologies.

In [2], consensus output regulation of network connected *MAS* is addressed. Every agent is represented by a nonlinear system, and has identical

dynamics. A consensus control design is proposed to ensure the convergence of all the agents to the same desired output trajectory by exploiting the internal model approach. The control design uses only the relative outputs of the agents, and it does not require the estimation of agent state variables.

The cooperative output regulation of linear *MAS* is considered in [17], where only a subset of subsystems can access the reference exogenous signal. The problem is solved by means of the design of distributed observers, and a dynamic full information distributed control scheme.

The simultaneous problem of consensus and trajectory tracking of linear multi-agent systems is considered in [11], where the dynamics of each agent is represented by a single-input single-output linear system and neither the structure nor the dimension of the representation of the agent dynamics are restricted to be the same. This problem was solved by a distributed control strategy, where the trajectory and the formation of the agents are achieved asymptotically even in the presence of switching communication topologies and smooth formation changes.

An approach to regulation of multi-agent linear systems is presented in [18], where the authors demonstrate zero tracking error under switching topologies using observers, but each agent observer is based on the exosystem output and on the states of the observers of its neighbors. In [21], the regulation of *MAS* with uncertainties on the matrices of the system is addressed. The results presented in that paper assure zero tracking error even when only a subset of the systems has access to the reference signal based on an internal model approach. The robust regulation of multi-agent linear systems considering uncertainties is addressed in [19]. The results obtained provide sufficient conditions for multi-agent zero tracking error under switched communication topologies based on a canonical internal model approach.

Problems related to multi-agent formation have been extensively undertaken, however as far as the authors know, the problem of solving simultaneously output regulation and output consensus of multi-agent switched linear systems has not been addressed before, and given the switched nature

of the agent dynamics, the application of previous results on regulation of *MAS* with switching topologies based on the canonical internal model is not straightforward and the existence of a solution cannot be guaranteed.

This paper presents the design of a distributed control law for a class of switched linear *MAS*, with switching communication topologies, where the output is the only information transmitted between the agents, and only a subset of them has access to the reference (output of the virtual agent). The virtual agent can be imagined as a transmitter providing the trajectory information for the group of agents or a real agent already regulated. The communication among the agents can be represented by a switching graph, containing a spanning tree. Given that the present approach focuses on output consensus, each of the agents is allowed to have different dynamics and state dimension.

In order to reach output consensus, the agents should have the same output dimension and it is considered that each agent has to follow the same reference provided by an exosystem, whose structure is known. Under such assumption, every agent is capable of computing the exosystem state by means of an observer. This approach is similar to the one presented in [18], however, the difference with the present work is that here, the observer depends only on the outputs of the neighbors of each agent. The proposed approach starts by computing a local feedback control law for all the linear systems that compose the switching dynamics of an agent. Such local feedback control is computed to ensure the existence of a common Lyapunov function to all linear systems of the agent, and this global Lyapunov function is used to guarantee the stability of the switched regulation and consensus error.

This work is organized as follows. Preliminary results used for the aims of this work are presented in Section 2, along with some basic notation. In Section 3, the problem of distributed output consensus and output regulation of multi-agent systems under switching communication topologies is defined. The main result and its development is shown in Section 4. An illustrative example is presented in Section 5, and finally, some conclusions are presented in Section 6.

2 Notation and Preliminaries

2.1 Graph Theory and Communication Topology

The exchange of information among the agents of a system is commonly described by a graph $\mathcal{G} = (\vartheta, \Phi, \mathcal{A})$, where ϑ is a set of nodes (agents), $\Phi \in \vartheta \times \vartheta$ is a set of edges that connect a node to another (self edges are not allowed), and $\mathcal{A} = [\alpha_{i,j}] \in \mathbb{R}^{N \times N}$ is its adjacency matrix containing positive weights describing the relationships among nodes. The Laplacian \mathcal{L} will be defined as $\mathcal{L} = \text{diag} \{ \sum_{j=1}^N \alpha_{1j} \cdots \sum_{j=1}^N \alpha_{Nj} \} - \mathcal{A}$. An edge $(\nu_i, \nu_j) \in \Phi$ means that node ν_j can get information from node ν_i . If an edge (ν_i, ν_j) is contained in Φ , this implies that the term $\alpha_{j,i}$ of the adjacency matrix is different from zero and vice versa, nevertheless, it does not imply that the edge (ν_j, ν_i) is also contained in Φ . The set of neighbors of node i will be denoted by $\Theta_i = \{ \nu_j : (\nu_j, \nu_i) \in \Phi, j = 1, \dots, N \}$ and by $\rho_i = |\Theta_i|$ its cardinality.

Let \mathcal{G} be a connected graph in which no cycles exist; then it is called a tree. A graph \mathcal{G} is said to have a spanning tree if every one of its nodes and a subset of its edges form a tree, which means that at least one of the nodes has a communication path to every other node.

In this work, given that switching communication topologies are considered, a switching graph becomes necessary. A switching graph will be defined as $\mathcal{G}_{\varrho_t} = (\vartheta, \Phi_{\varrho_t}, \mathcal{A}_{\varrho_t})$, where $\varrho_t : [t_0, \infty) \rightarrow \{1, \dots, \gamma\}$ is the switching signal that determines the communication topology at time t . It will be considered that the set of nodes ϑ is constant and that the condition $(\nu_i, \nu_j) \in \Phi_{\varrho_t} \Rightarrow (\nu_j, \nu_i) \notin \Phi_{\varrho_t}$ is met. The set of neighbors of node i at time t , $\Theta_{i_{\varrho_t}}$, will be defined analogously as $\Theta_{i_{\varrho_t}} = \{ \nu_j : (\nu_j, \nu_i) \in \Phi_{\varrho_t}, j = 1, \dots, N \}$ and $\rho_{i_{\varrho_t}} = |\Theta_{i_{\varrho_t}}|$ its cardinality.

2.2 Switched Linear Systems

A Switched Linear System (SLS) $\mathcal{V} = \langle \mathcal{F}, \sigma_t \rangle$ is a hybrid dynamical system where $\mathcal{F} =$

$\{ \Sigma_1, \Sigma_2, \dots, \Sigma_k \}$ is a collection of linear systems of the form

$$\Sigma_{\sigma_t} \begin{cases} \dot{x}(t) = A_{\sigma_t} x(t) + B_{\sigma_t} u(t); & x(t_0) = x_0, \\ y(t) = Cx(t), \end{cases} \quad (1)$$

where $\sigma_t : [t_0, \infty) \rightarrow \{1, \dots, k\}$ is the switching signal that determines the evolving linear dynamics, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^q$, are the state, input and output variables, respectively, and A_{σ_t} , B_{σ_t} and C , $\sigma_t \in \{1, \dots, k\}$ are constant matrices of appropriate dimensions.

Remark 1 In systems as (1), it is often considered that $y(t) = C_{\sigma_t} x(t)$, but for the objective of this work we restrict ourselves to consider systems of the form (1) with $C_{\sigma_t} = C$ for all $\sigma_t \in \{1, \dots, k\}$.

2.2.1 Lyapunov Stability

The switched linear system (1) under a state feedback control

$$u(t) = -K_{\sigma_t} x(t), \quad (2)$$

is *uniformly asymptotically stable* if there exist a positive constant g and a class \mathcal{KL} function f such that for all switching signals σ_t the solutions of (1) with $|x(0)| \leq g$ satisfy

$$|x(t)| \leq f(|x(0)|, t), \quad \forall t \geq 0. \quad (3)$$

If the inequality (3) is valid for all switching signals and all initial conditions, the system (1) under a state feedback control (2) is *globally uniformly asymptotically stable (GUAS)*.

A positive definite continuously differentiable function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is a common Lyapunov function for the family of systems \mathcal{F} if there exists a positive definite continuous function $Q : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\frac{\partial V}{\partial x} (A_{\sigma_t} - B_{\sigma_t} K_{\sigma_t}) x(t) \leq -Q(x(t)), \quad \forall x(t), \forall \sigma_t. \quad (4)$$

Then, the following result on stability can be stated.

Lemma 1 (Theorem 2.1 [8]) *If all systems in the family \mathcal{F} with control (2) share a radially unbounded common Lyapunov function $V(x(t))$, then the switched system (1) is globally uniformly asymptotically stable (GUAS).*

This result assures the stability of the *SLS* system under any switching sequence. It is important to mention that if a common Lyapunov function for the *SLS* is not found, it is not enough proof to assert that a *SLS* is unstable.

The following result from [13] will be used afterwards.

Lemma 2 *If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ and matrices $Z_{\sigma_t} \in \mathbb{R}^{m \times n}$, $\sigma_t = \{1, \dots, k\}$, such that*

$$A_{\sigma_t}P + PA_{\sigma_t}^T - B_{\sigma_t}Z_{\sigma_t} - Z_{\sigma_t}^TB_{\sigma_t}^T < 0,$$

then the switched control law (2) with

$$K_{\sigma_t} = Z_{\sigma_t}P^{-1},$$

assures the closed-loop stability of the switched system (1).

In this way, it is possible to state the problem of finding a common Lyapunov function for a *SLS* system as a Linear Matrix Inequality (*LMI*) problem to compute a switched state feedback.

2.2.2 Output Regulation

Let the system

$$\dot{\omega}(t) = S\omega(t), \quad (5)$$

where $\omega(t) \in \mathbb{R}^p$ and $S \in \mathbb{R}^{p \times p}$, be an exosystem from which a reference $y_r(t) = R\omega(t)$ will be defined for an *SLS* system and consider the input to the *SLS* as

$$u(t) = -K_{\sigma_t}x(t) + \Gamma_{\sigma_t}\omega(t). \quad (6)$$

Then, from [10], the problem of Output Regulation via Full Information for *SLS* (*ORFI*) consists in, given A_{σ_t} , B_{σ_t} , C , R , S and having full access to $x(t)$, and $\omega(t)$, finding a feedback control law of the form (6) such that

- i) system (1) with control (2) is asymptotically stable under arbitrary switching laws, and
- ii) for each initial condition x_0 , the solution $x(t)$ of system (1) with (6) is such that $y(t) - y_r(t)$ tends to zero uniformly under arbitrary switching laws.

Consider the following hypotheses.

Hypothesis 1 ($\mathcal{H}1$) *System (5) is antistable, i.e. all the eigenvalues of S have nonnegative real part.*

Hence, it is guaranteed that the reference will never tend to zero and will be continuously changing.

Hypothesis 2 ($\mathcal{H}2$) *The pair (R, S) is detectable.*

Hypothesis 3 ($\mathcal{H}3$)

a) *System (1) with control (2) has a common Lyapunov function.*

b) *There exists a solution matrix Π_j to the following equation*

$$\begin{aligned} \Pi_{\sigma_t}S &= (A_{\sigma_t} - B_{\sigma_t}K_{\sigma_t})\Pi_{\sigma_t} + B_{\sigma_t}\Gamma_j \\ 0 &= C\Pi_{\sigma_t} - R \end{aligned} \quad \sigma_t = 1, 2, \dots, k.$$

Hypothesis 4 ($\mathcal{H}4$) *It will be considered that the evolving dynamics of system (1) and its switching instants are always known.*

Lemma 3 ([10], Theorem 1) *Assume ($\mathcal{H}1$) and ($\mathcal{H}3$). Then the ORFI of switched system (1) is solvable only if $\forall h, \sigma_t$, $h, \sigma_t = 1, 2, \dots, k$, the following condition holds:*

$$\begin{aligned} C(A_l - B_lK_l)^m(\Pi_h - \Pi_{\sigma_t}) &= 0, \\ m &= 0, 1, \dots, s \quad l = h, \sigma_t, \end{aligned}$$

where s is the index of the pair $(C, A_{\sigma_t} - B_{\sigma_t}K_{\sigma_t})$.

A more restrictive condition is given if $(A_{\sigma_t} - B_{\sigma_t}K_{\sigma_t}, C)$ is observable, as in the following result.

Lemma 4 ([10], Theorem 2) *Assume ($\mathcal{H}1$), ($\mathcal{H}3$), and that $(A_{\sigma_t} - B_{\sigma_t}K_{\sigma_t}, C)$, $\sigma_t = 1, 2, \dots, k$ is observable. Then the ORFI of (1) is solvable if and only if $\Pi_h = \Pi_{\sigma_t}$, for $h, \sigma_t = 1, 2, \dots, k$.*

From Lemma 3, it is clear that under Hypotheses ($\mathcal{H}1$) and ($\mathcal{H}3$) a sufficient condition for the problem of *ORFI* to be solvable is the following: if the subset of observable states of every subsystem $\Sigma_h \in \mathcal{F}$ is the same, the elements of Π_h corresponding to observable states have to be equal to those of Π_{σ_t} for $h, \sigma_t = 1, 2, \dots, k$.

A class of *SLS* systems that satisfies the previous condition is the class of *SLS* systems of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & 0 \\ A_{3,\sigma_t} & A_{4,\sigma_t} & 0 \\ A_{5,\sigma_t} & A_{6,\sigma_t} & A_{7,\sigma_t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ B_{1,\sigma_t} \\ B_{2,\sigma_t} \end{bmatrix} u$$

$$y(t) = [C_1 \ 0] \begin{bmatrix} x^\eta(t) \\ x_3(t) \end{bmatrix}, \quad (7)$$

for $\sigma_t = 1, 2, \dots, k$, where $x_1(t) \in \mathbb{R}^{\eta-q}$, $x_2(t) \in \mathbb{R}^q$, $x_3(t) \in \mathbb{R}^{n-\eta}$, $u(t) \in \mathbb{R}^m$, $q \leq m$, B_{1,σ_t} is a full row rank matrix. The submatrices A_1 and A_2 represent the non-switching part of the system.

Additionally, let

$$A_{\sigma_t}^\eta = \begin{bmatrix} A_1 & A_2 \\ A_{3,\sigma_t} & A_{4,\sigma_t} \end{bmatrix}, \quad B_{\sigma_t}^\eta = \begin{bmatrix} 0 \\ B_{1,\sigma_t} \end{bmatrix},$$

$$x^\eta(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

the subsystem

$$\dot{x}^\eta(t) = A_{\sigma_t}^\eta x^\eta(t) + B_{\sigma_t}^\eta u(t), \quad (8)$$

must be controllable.

The values of Π_i corresponding to the observable part of the linear subsystems are determined by their non-switching part, therefore they are equal for every linear subsystem. The subsystem $\dot{x}_3(t) = A_{7,\sigma_t} x_3(t) + B_{2,\sigma_t} u(t)$ represents the unobservable part of the system.

The previous systems (7) are introduced to be used in the simulation example. It is worth noting that the results presented are not restricted to such systems.

3 Problem Statement

Consider a MAS consisting of N agents with different SLS dynamics of class (7) described by

$$\begin{aligned} \dot{x}_i(t) &= A_{i,\sigma_{it}} x_i(t) + B_{i,\sigma_{it}} u_i(t) & i &= 1, \dots, N \\ y_i(t) &= C_i x_i(t) & \sigma_{it} &= 1, \dots, k_i \end{aligned} \quad (9)$$

where, $x_i(t) \in \mathbb{R}^{n_i}$ is the state, $u_i(t) \in \mathbb{R}^{m_i}$ the control, and $y_i(t) \in \mathbb{R}^q$ the output of the i -th agent,

and a virtual reference agent whose dynamics is represented by the exosystem (5) with output

$$y_0(t) = R\omega(t) \quad (10)$$

where $R \in \mathbb{R}^{q \times p}$.

Remark 2 It is considered that $\text{rank}\{C_i\} = q, \forall i$.

Let \mathcal{G}_{ϱ_t} be a switching graph representing communication among the SLS agents of system (9) together with the exosystem (5) where $\varrho_t : [t_0, \infty) \rightarrow \{1, \dots, v\}$ is the switching signal that determines the evolving communication topology, then the following hypothesis is assumed.

Hypothesis 5 (H5) The graph \mathcal{G}_{ϱ_t} does not contain loops, $(\nu_i, \nu_j) \in \Phi_{\varrho_t} \Rightarrow (\nu_j, \nu_i) \notin \Phi_{\varrho_t}$, and incorporates a spanning tree with root on $\nu_{r_{\varrho_t}}$, the reference agent for a given ϱ_t .

For a MAS, the consensus is achieved by the states of the agent if all of them converge to the same value. In this work only the output is considered in the consensus, i.e., it is required that $\lim_{t \rightarrow \infty} |y_i - y_j| = 0$ for all $i, j = 0, 1, \dots, N$. Additionally, the output regulation deals with the problem of a system following an exosystem where $\lim_{t \rightarrow \infty} |Cx(t) - R\omega(t)| = 0$. Now, we define the switching i -th agent output consensus and output regulation error as

$$\begin{aligned} \zeta_{i_{\varrho_t}}(t) &= \sum_{j=1}^N \alpha_{i,j,\varrho_t} [(y_i(t) - y_j(t))] \\ &\quad + \alpha_{i,0,\varrho_t} [(y_i(t) - y_0(t))]. \end{aligned} \quad (11)$$

Note that the weight α_{i,j,ϱ_t} is zero if agent i does not receive information from agent j , similarly, the weight $\alpha_{i,0,\varrho_t}$ applies to the difference between the i -th agent output and the virtual agent (reference) output (10). Remember that these weights depend on ϱ_t (the evolving communication topology).

Thus, the problem of distributed output regulation and output consensus of SLS multi-agent systems under switching interaction topologies consists in obtaining a distributed control law such that every agent output consensus and tracking error $\zeta_{i_{\varrho_t}}(t)$ tends to zero asymptotically under any communication switching sequence of ϱ_t .

This problem is addressed and solved in the next section.

4 Distributed Output Regulation and Output Consensus Control

For ease of notation and without loss of generality, consider the controllable part (8) of N SLS agents in the form (7) as in (9).

Given that only a subset of the SLS agents will have access to reference (10) and none of them to the exosystem state (5), the following observers are proposed:

$$\dot{\hat{\omega}}_i(t) = S\hat{\omega}_i(t) + \frac{\kappa_i}{\rho_{i_{\Theta_t}}} \left(\sum_{j \in \Theta_{i_{\Theta_t}}} y_j(t) - \rho_{i_{\Theta_t}} R\hat{\omega}_i(t) \right), \quad (12)$$

where κ_i is such that the matrix $(S - \kappa_i R)$ is Hurwitz. Now, set the distributed output regulation control law as

$$u_i(t) = -K_{i,\sigma_{it}} x_i(t) + \Gamma_{i,\sigma_{it}} \hat{\omega}_i(t), \quad (13)$$

for $\sigma_{it} \in \{1, \dots, k_i\}$. Note that control (13) is of form (6) and depends on the exosystem observer state $\hat{\omega}_i(t)$.

Define the i -th observer error as

$$\tilde{\omega}_i(t) = \omega(t) - \hat{\omega}_i(t), \quad (14)$$

hence, its dynamics will be described by

$$\dot{\tilde{\omega}}_i(t) = S\tilde{\omega}_i(t) - \frac{\kappa_i}{\rho_{i_{\Theta_t}}} \left(\sum_{j \in \Theta_{i_{\Theta_t}}} y_j(t) - \rho_{i_{\Theta_t}} R\hat{\omega}_i(t) \right). \quad (15)$$

Consider that Hypothesis (H3) is met, then the i -th agent regulation error can be defined as

$$e_i(t) = x_i(t) - \Pi_i \hat{\omega}_i(t), \quad (16)$$

and its dynamics is given by

$$\begin{aligned} \dot{e}_i(t) &= (A_{i,\sigma_{it}} - B_{i,\sigma_{it}} K_{i,\sigma_{it}}) e_i(t) \\ &\quad - \Pi_i \frac{\kappa_i}{\rho_{i_{\Theta_t}}} \left(\sum_{j \in \Theta_{i_{\Theta_t}}} y_j(t) - \rho_{i_{\Theta_t}} R\hat{\omega}_i(t) \right). \end{aligned} \quad (17)$$

From (14) and (16), we have that

$$\begin{aligned} y_i(t) &= C_i x_i(t) \\ &= C_i (e_i(t) + \Pi_i \hat{\omega}_i(t)) \\ &= C_i (e_i(t) + \Pi_i (\omega(t) - \tilde{\omega}_i(t))) \\ &= C_i e_i(t) + R (\omega(t) - \tilde{\omega}_i(t)). \end{aligned} \quad (18)$$

In order to save space, let us work with the second term in (15) substituting y_j as in (18):

$$\begin{aligned} \sum_{j \in \Theta_{i_{\Theta_t}}} y_j(t) - \rho_{i_{\Theta_t}} R\hat{\omega}_i(t) &= -\rho_{i_{\Theta_t}} R\hat{\omega}_i(t) \\ &\quad + \sum_{j \in \Theta_{i_{\Theta_t}}} \{R(\omega(t) - \tilde{\omega}_j(t)) + C_j e_j(t)\}. \end{aligned} \quad (19)$$

Hence,

$$\begin{aligned} \sum_{j \in \Theta_{i_{\Theta_t}}} y_j(t) - \rho_{i_{\Theta_t}} R\hat{\omega}_i(t) &= -\rho_{i_{\Theta_t}} R\hat{\omega}_i(t) \\ &\quad + \rho_{i_{\Theta_t}} R\omega(t) + \sum_{j \in \Theta_{i_{\Theta_t}}} (C_j e_j(t) - R\tilde{\omega}_j(t)) \end{aligned} \quad (20)$$

and

$$\begin{aligned} \sum_{j \in \Theta_{i_{\Theta_t}}} y_j(t) - \rho_{i_{\Theta_t}} R\hat{\omega}_i(t) &= \rho_{i_{\Theta_t}} R\tilde{\omega}_i(t) \\ &\quad + \sum_{j \in \Theta_{i_{\Theta_t}}} (C_j e_j(t) - R\tilde{\omega}_j(t)). \end{aligned} \quad (21)$$

Using (21) we can restate the i -th observer and regulation error dynamics (15), (17) as

$$\begin{aligned} \dot{\tilde{\omega}}_i(t) &= \frac{\kappa_i}{\rho_{i_{\Theta_t}}} \left(\sum_{j \in \Theta_{i_{\Theta_t}}} (R\tilde{\omega}_j(t) - C_j e_j(t)) \right) \\ &\quad + (S - \kappa_i R) \tilde{\omega}_i(t) \end{aligned} \quad (22)$$

$$\begin{aligned} \dot{e}_i(t) &= \Pi_i \frac{\kappa_i}{\rho_{i_{\Theta_t}}} \sum_{j \in \Theta_{i_{\Theta_t}}} (R\tilde{\omega}_j(t) - C_j e_j(t)) \\ &\quad + (A_{i,\sigma_{it}} - B_{i,\sigma_{it}} K_{i,\sigma_{it}}) e_i(t) - \Pi_i \kappa_i R\tilde{\omega}_i(t). \end{aligned} \quad (23)$$

Define the i -th agent observer-regulation error as

$$\xi_i(t) = \begin{bmatrix} \tilde{\omega}_i(t) \\ e_i(t) \end{bmatrix}, \quad (24)$$

and SLS multi-agent observer-regulation error as

$$\xi(t) = [\xi_1^T(t) \quad \xi_2^T(t) \quad \dots \quad \xi_N^T(t)]^T. \quad (25)$$

Consider that Hypothesis (H5) is met, then given any communication topology \mathcal{G}_{γ_0} , there exists an order for the systems (O_{γ_0}) , such that the Laplacian \mathcal{L}_{γ_0} is a lower triangular matrix. Thus, reordering the agents according to the order O_{γ_0} , the SLS multi-agent observer-regulation error dynamics will have a block lower triangular form

$$\dot{\xi}_{\gamma_0}(t) = \bar{A}_{\gamma_0} \xi_{\gamma_0}(t) \quad (26)$$

where the matrix \bar{A}_{γ_0} can be described as follows:

$$\bar{A}_{\gamma_0} = \begin{bmatrix} \Omega_{1,\sigma_{1t}} & 0 & \cdots & 0 \\ \beta_{2,1} & \Omega_{2,\sigma_{2t}} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \beta_{N,1} & \beta_{N,2} & \cdots & \Omega_{N,\sigma_{Nt}} \end{bmatrix} \quad (27)$$

with indexes according to the order O_{γ_0} and switched matrix blocks

$$\Omega_{i,\sigma_{it}} = \begin{bmatrix} S - \kappa_i R & 0 \\ -\Pi_i \kappa_i R & A_{i,\sigma_{it}} - B_{i,\sigma_{it}} K_{i,\sigma_{it}} \end{bmatrix}, \quad (28)$$

on its diagonal, where $\xi_{\varrho_t}(t) = \mathcal{T}_{\varrho_t} \xi(t)$ for $\varrho_t = \gamma_0$ and $\mathcal{T}_{\varrho_t} \in \mathbb{R}^{\Sigma n_i \times \Sigma n_i}$ is a transformation matrix, which changes the order of the agents.

Hence, it is clear that the SLS multi-agent observer-regulation error will have switched linear dynamics

$$\dot{\xi}(t) = \tilde{A}_{\varrho_t} \xi(t), \quad (29)$$

where $\tilde{A}_{\varrho_t} \in \mathbb{R}^{\Sigma n_i \times \Sigma n_i}$.

Now the main result of this work can be stated.

Theorem 1 *Let N SLS agents be of form (7), with a switching communication topology such that Hypothesis (H5) is met. Consider also, that Hypotheses (H1)-(H4) are met, then the distributed output regulation and output consensus of the SLS multi-agent system can be solved by a control law of form (13).*

Proof: Consider an SLS multi-agent system with agents of the form (7), an exosystem (5) with output (10), and control law (13).

Then, under Hypothesis (H5) and a given order O_{γ_0} , the corresponding SLS multi-agent observer-regulation error will have form (26). Given Hypothesis (H3), the switched block matrices $\Omega_{i,\sigma_{it}}$ are such that there exist positive definite matrices P_i and $Q_{i,\sigma_{it}}$, for which the equations

$$\Omega_{i,\sigma_{it}}^T P_i + P_i \Omega_{i,\sigma_{it}} = -Q_{i,\sigma_{it}} \quad (30)$$

are met.

A common Lyapunov function for system (26) is selected as

$$V(t) = \sum_{j=1}^N V_j(t), \quad (31)$$

where

$$V_i(t) = \xi_i(t)^T P_i \xi_i(t). \quad (32)$$

Consider $V_1(t)$, whose derivative

$$\dot{V}_1(t) = -\xi_1(t)^T Q_{1,\sigma_{1t}} \xi_1(t), \quad (33)$$

is clearly negative, independent from the value of $\xi_1(t)$. Thus, the observer-regulation error $\xi_1(t)$ is stable. For $V_2(t)$ its derivative is

$$\begin{aligned} \dot{V}_2(t) = & -\xi_2(t)^T Q_{2,\sigma_{2t}} \xi_2(t) + \xi_1(t)^T \beta_{2,1}^T P_2 \xi_2(t) \\ & + \xi_2(t)^T P_2 \beta_{2,1} \xi_1(t), \end{aligned} \quad (34)$$

hence, given the stability of $\xi_1(t)$, the terms with $\beta_{2,1}$ will become zero and the derivative of $V_2(t)$ will be negative, therefore, the observer-regulation error $\xi_2(t)$ is stable. The same reasoning can be applied to the rest of the $V_i(t)$'s:

$$\begin{aligned} \dot{V}_i(t) = & \sum_{j \in \Theta_{i,\varrho_t}} \xi_j(t)^T \beta_{\sigma_{it},i}^T P_{\sigma_{it}} \xi_{\sigma_{it}}(t) \\ & + \sum_{j \in \Theta_{i,\varrho_t}} \xi_{\sigma_{it}}(t)^T P_{\sigma_{it}} \beta_{\sigma_{it},i} \xi_j(t) \\ & - \xi_i(t)^T Q_{i,\sigma_{it}} \xi_i(t). \end{aligned} \quad (35)$$

In this way, we conclude that the derivative of the common Lyapunov function (31) is negative, and as a result, the observer-regulation error (26) is stable.

Given that the diagonal blocks are always the same for a given agent, the previous procedure can be carried out for any order O_{ϱ_t} , then function (31) is a common Lyapunov function for the observer-regulation error system (29), which is thus ultimately bounded.

Finally, given the stability of the observer-regulation error (29), the following holds in steady state

$$\hat{\omega}_i(t) = \omega(t), \quad (36)$$

$$x_i(t) = \Pi_i \omega(t), \quad (37)$$

$$y_i(t) = R\omega(t), \quad (38)$$

thus, the i -th output consensus and tracking error $\zeta_{i,\varrho_t}(t) = 0$ for $i = 1, 2, \dots, N$. ■

5 Example

The following example illustrates the results presented in this paper. Consider 3 agents with the dynamics

$$\dot{x}_i = A_{i,\sigma_{it}} x_i + B_{i,\sigma_{it}} u_i,$$

$$y_i = C_i x_i,$$

where

$$A_{1,1} = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 9 & -2 & 4 & 0 \\ 0 & 0 & 1 & -3 \\ 7 & 4 & -1 & -1 \end{bmatrix}, \quad B_{1,1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 7 & 0 \end{bmatrix},$$

$$A_{1,2} = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 9 & -2 & 4 & 0 \\ 4 & -2 & 5 & 1 \\ 6 & 2 & 5 & -3 \end{bmatrix}, \quad B_{1,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 4 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix}, \quad A_{2,1} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -1 & 0 \\ 4 & 7 & -7 \end{bmatrix},$$

$$A_{2,2} = \begin{bmatrix} 10 & -1 & 0 \\ -3 & 1 & 0 \\ 9 & -3 & -1 \end{bmatrix}, \quad A_{2,3} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix},$$

$$B_{2,1} = \begin{bmatrix} 0 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -3 & 0 & 0 & 5 \\ 3 & 2 & 0 & 3 \\ 1 & 4 & -2 & 0 \\ 0 & -2 & -2 & -1 \end{bmatrix}, \quad B_{3,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix},$$

$$B_{3,2} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 3 \end{bmatrix},$$

with initial conditions

$$x_1(0) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}, \quad x_2(0) = \begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix}, \quad x_3(0) = \begin{bmatrix} 5 \\ -2 \\ -4 \\ 2 \end{bmatrix}.$$

The communication topology of the multi-agent system, including the systems above, is considered to be switching every 250 milliseconds on a

cyclic sequence $\{1, 2, 3, 4, 1, \dots\}$ according to the following Laplacian matrices

$$\mathcal{L}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \quad \mathcal{L}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\mathcal{L}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{L}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Note that the first row corresponds to the reference node, hence, the minus one in the first column indicates which agent receives the virtual agent's output.

Let the reference exosystem be

$$\dot{\omega} = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega,$$

consider the initial conditions

$$\omega(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

and take the reference matrix as

$$R = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

From Lemma 2, the stabilizing feedback matrices calculated for the system by a *LMI* as in [13] are

$$K_{1,1} = \begin{bmatrix} 1.14 & 0.63 & -0.05 & 0.63 \\ 6.31 & 1.91 & 6.31 & 3.29 \end{bmatrix},$$

$$K_{1,2} = \begin{bmatrix} 4.47 & 0.62 & 4.15 & 2.81 \\ 6.89 & 2.43 & 5.53 & 2.37 \end{bmatrix},$$

$$K_{2,1} = \begin{bmatrix} -4 & 2.33 & 0 \\ 1 & 0.33 & 0 \end{bmatrix},$$

$$K_{2,2} = \begin{bmatrix} -11.66 & 5.66 & 0 \\ 4.33 & -0.33 & 0 \end{bmatrix},$$

$$K_{2,3} = \begin{bmatrix} -1.66 & 4 & 0 \\ 1.33 & 0 & 0 \end{bmatrix},$$

$$K_{3,1} = \begin{bmatrix} 11.05 & 22.59 & 2.80 & 12.38 \\ -22.54 & -50 & -9.85 & -19.83 \end{bmatrix},$$

$$K_{3,2} = \begin{bmatrix} 12.10 & 32.63 & 11.27 & 2.68 \\ -0.49 & -4.94 & -4.23 & 4.88 \end{bmatrix},$$

and the corresponding Γ matrices are given by

$$\Gamma_{1,1} = \begin{bmatrix} 1.41 & -0.56 & 0.72 \\ -3.77 & 1.83 & -1.66 \end{bmatrix},$$

$$\Gamma_{1,2} = \begin{bmatrix} -1.54 & 0.78 & -0.65 \\ 9.83 & -3.96 & 5.03 \end{bmatrix},$$

$$\Gamma_2 = \begin{bmatrix} -2.33 & 5.66 & 1 \\ 1.66 & -1.83 & 0.5 \end{bmatrix},$$

$$\Gamma_{3,1} = \begin{bmatrix} 4.73 & 5.92 & -22.35 \\ -16.87 & -5.96 & 44.69 \end{bmatrix},$$

$$\Gamma_{3,2} = \begin{bmatrix} 19.31 & -5.56 & -22.94 \\ -7.32 & 5.79 & 0.17 \end{bmatrix}.$$

The references and the trajectories of the outputs of the multi-agent system obtained in simulation are shown in Fig. 1 and Fig. 2. As it can be seen, the outputs $y_{i,1}$ and $y_{i,2}$ follow the same trajectory and converge to the same reference, respectively. The exosystem ω is in this case composed by a sine function ω_1 , a cosine function ω_2 , and a constant $\omega_3 = 1$. The value of the reference is attained by every output.

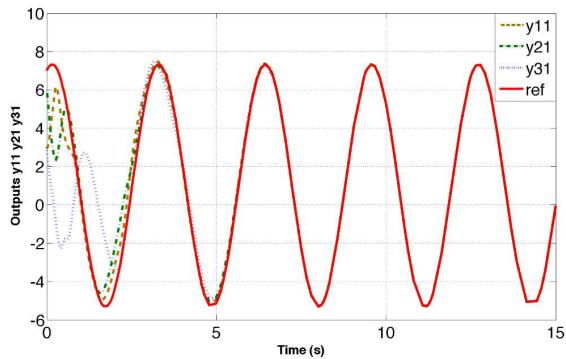


Fig. 1. Outputs reaching formation and tracking

Fig. 3 displays the behavior of the regulation error (17), while Fig. 4 displays the regulation error with respect to the state of the exosystem $error_i = x_i - \Pi_i \hat{\omega}$. From these figures, it can

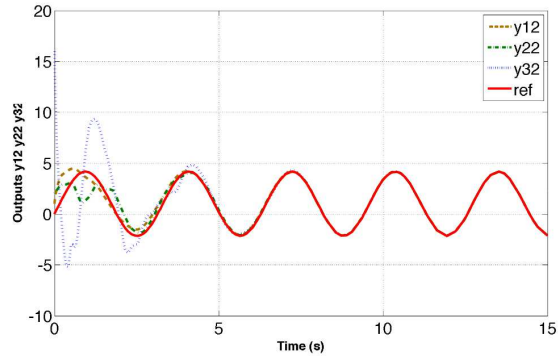


Fig. 2. Outputs reaching formation and tracking

be observed that most of the errors tend to zero except the dashed lines on errors of system 2 that remain bounded.

The dashed lines on the errors of system 2 do not decay to zero because the corresponding state does not share the same terms on the corresponding matrix Π . But this error does not affect the regulation given that such a state is not observable.

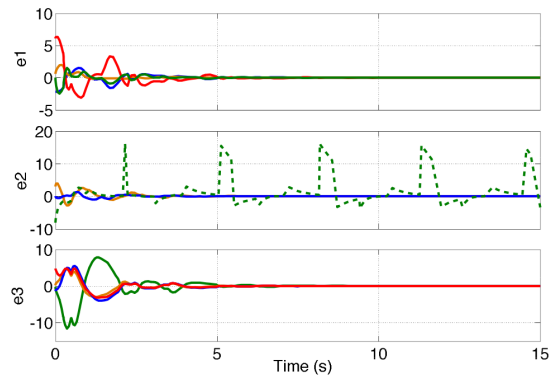


Fig. 3. Output regulation and consensus error $e_i = x_i - \Pi_i \hat{\omega}$.

6 Conclusions

This work presented the design of a distributed control law for output consensus and output regulation for a group of SLS agents of class (7) under switched communication topologies.

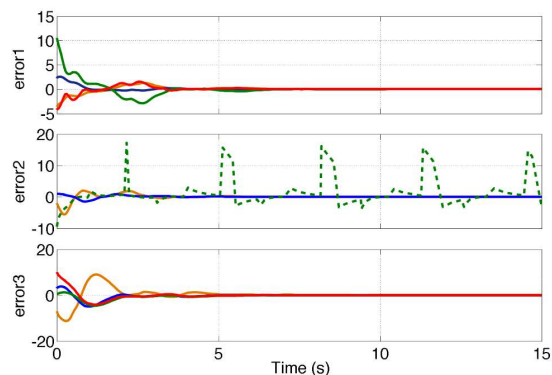


Fig. 4. Output regulation and consensus error $error_i = x_i - \Pi_i \omega$.

The proposed strategy achieves the output regulation and consensus of the SLS multi-agent system, under switching communication topologies, preserving a desired formation even when only a subset of the agents has access to the reference, and none of them have access to the state of the exosystem.

The observer-regulation error stability is guaranteed by properly selecting the stabilizing error feedback $K_{i,\sigma_{it}}$ to assure the existence of a common Lyapunov function for the i -th system.

The results of this paper can be extended by dropping assumption (H4) applying the results presented in [5]. Also, a study of other control techniques in order to deal with discrete communication with delays can be performed.

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