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Fast Convergence of the Two Dimensional Discrete Shearlet Transform

Jaime Navarro Fuentes[1], Oscar Herrera Alcántara[2]

[1] Universidad Autónoma Metropolitana,
Departamento de Ciencias Básicas,
Mexico

[2] Universidad Autónoma Metropolitana,
Departamento de Sistemas,
Mexico

jnfu@azc.uam.mx, oha@azc.uam.mx

Abstract. The 2D discrete shearlet transform for a given function f in $L^2(\mathbb{R}^2)$ has been defined through dilation, shear and translation parameters in such a way that the continuity of f at $(0,0)$ can be studied by means of the convergence of the discrete shearlet transform as the dilation parameter converges to zero. Computer experiments illustrate this property by detecting edges in images that correspond to discontinuities.

Keywords. Shearlets, tight frames, continuity, image processing.

1 Introduction

Wavelets are used to approximate smooth functions with point singularities. In higher dimensions wavelets detect the point singularities but do not give information about the directions where these singularities occur, specially discontinuities along lines or curves [1]. This problem has been studied by different class of wavelets like shearlets, which are obtained by means of dilations, translations and shear parameters [2].

Shearlets have been defined in such a way that they provide not only the point singularities but also the directions of these singularities. That is, the shearlet representation is more effective than the wavelet representation for the analysis and processing of multidimensional data [3].

Wavelets do not detect, with good approximation, the geometry of images specially with edges. Hence, shearlets have been defined under a similar framework of wavelets, but with composite dilations. Thus, shearlets have been a very useful tool to obtain better theoretical results and applications than the ones obtained from the wavelet theory [1].

In one dimension, for a given admissible function $h \in L^2(\mathbb{R})$, the continuous wavelet transform for a given signal $f \in L^2(\mathbb{R})$ is defined as [4]:

$$(L_h f)(a,b) = \int_{\mathbb{R}} f(x) \frac{1}{\sqrt{|a|}} \overline{h}\left(\frac{x-b}{a}\right) dx,$$

where $a \neq 0$ and $b \in \mathbb{R}$.

In this case, $h \in L^2(\mathbb{R})$ is admissible if:

$$0 < C_h := \int_{\mathbb{R}} |\widehat{h}(k)|^2 \frac{1}{|k|} \, dk < \infty.$$

The discrete wavelet transform is obtained by considering discrete values for the dilation parameter a and the translation parameter b. That is, if $a = a_0^m$ and $b = nb_0a_0^m$, with $a_0 > 1$ and $b_0 > 0$, are fixed and $m, n \in \mathbb{Z}$, the discrete wavelet transform of $f \in L^2(\mathbb{R})$ with respect to the admissible function $h \in L^2(\mathbb{R})$ is given by [5]:

$$\begin{aligned}&(L_h f)(a_0^m, nb_0a_0^m)\\ &= \int_{\mathbb{R}} f(x) \frac{1}{\sqrt{a_0^m}} \overline{h}\left(\frac{x - nb_0a_0^m}{a_0^m}\right) dx.\end{aligned}$$

The dilation and translation operators $J_{a_0^m}$ and $T_{nb_0a_0^m}$ are defined respectively as:

$$(J_{a_0^m}h)(x) = \frac{1}{\sqrt{a_0^m}} h\left(\frac{x}{a_0^m}\right),$$

$$(T_{nb_0a_0^m}h)(x) = h(x - nb_0a_0^m),$$

then the discrete wavelet transform can be written as:

$$(L_hf)(a_0^m, nb_0a_0^m) = \left\langle f, T_{nb_0a_0^m}J_{a_0^m}h\right\rangle.$$

Moreover, if $(T_{nb_0a_0^m}J_{a_0^m}h)_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ is a tight frame in $L^2(\mathbb{R})$, then for all $f \in L^2(\mathbb{R})$ there is a positive constant C such that the inversion formula is given by [6]:

$$f = \frac{1}{C}\sum_{m,n\in\mathbb{Z}} \left\langle f, T_{nb_0a_0^m}J_{a_0^m}h\right\rangle T_{nb_0a_0^m}J_{a_0^m}h.$$

For n dimensions, the dilation parameter a is in $\mathbb{R}^+$, and the translation parameter b is in $\mathbb{R}^n$, where $b = (b_1, b_2, \ldots, b_n)$ with $b_i > 0,\ i = 1, 2, \ldots n$.

In this case, for $a > 1$ and $j = (j_1, j_2, \ldots, j_n)$ in $\mathbb{Z}^n$, the $n \times n$ matrix $M(a^j)$ is defined as:

$$M(a^j) = diag(a^{j_1}, a^{j_2}, \ldots, a^{j_n}),$$

and for $b = (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n_+$, the $n \times n$ matrix $M(b)$ is defined as :

$$M(b) = diag(b_1, b_2, \ldots, b_n).$$

Now, for $a > 1$ and $b = (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n_+$, the dilation and translation operators are defined respectively as:

$$(J_{M(a^j)}h)(x) = \frac{1}{\sqrt{M(a^j)}} h\left(M^{-1}(a^j)x\right),$$

where $x \in \mathbb{R}^n$, $j \in \mathbb{Z}^n$, and

$$(T_{M^{-1}(a^j)M(b)k}h)(x) = h(x - M^{-1}(a^j)M(b)k),$$

where $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}^n$. Hence, the discrete wavelet transform in n dimensions for $f \in L^2(\mathbb{R}^n)$ with respect to a radially symmetric admissible function $h \in L^2(\mathbb{R}^n)$, is defined as [7]:

$$\begin{aligned}&(L_hf)(M^{-1}(a^j), M^{-1}(a^j)M(b)k)\\ &= \left\langle f, T_{M^{-1}(a^j)M(b)k}J_{M(a^j)}h\right\rangle.\end{aligned}$$

In this case, $h \in L^2(\mathbb{R}^n)$ is admissible if:

$$\int_0^\infty |\eta(k)|^2 \frac{1}{k}\, dk < \infty, \quad \text{where } \widehat{h}(y) = \eta(|y|).$$

Moreover, if $(T_{M^{-1}(a^j)M(b)k}J_{M(a^j)}h)_{(j,k)\in\mathbb{Z}^n\times\mathbb{Z}^n}$ is a tight frame in $L^2(\mathbb{R}^n)$, then for all $f \in L^2(\mathbb{R}^n)$ there is a positive constant C such that [8]:

$$\begin{aligned}f =& \frac{1}{C}\sum_{j,k\in\mathbb{Z}^n} \left\langle f, T_{M^{-1}(a^j)M(b)k}J_{M(a^j)}h\right\rangle\\ &\cdot T_{M^{-1}(a^j)M(b)k}J_{M(a^j)}h.\end{aligned}$$

In this paper, in Section 2 the continuous shearlet transform is studied in two dimensions and in Section 3 discrete values for the dilation, shear and translation parameters are taken to obtain the discrete shearlet transform following the same idea from the definition of the discrete wavelet transform and then, in Sections 4and 5 a tight frame is considered in $L^2(\mathbb{R}^2)$ to analyze the continuity of a given function $f \in L^2(\mathbb{R}^2)$ under the hypothesis of the fast convergence of its discrete shearlet transform. In Section 6, an example is given to illustrate the results by means of a computational experiment, and finally in Section 7 conclusions are presented.

2 Notations and Definitions

First, an overview of the continuous shearlet transform in two dimensions is given. In this case, the shearlet transform is defined with respect to: dilations, shears and translations parameters. That is, the following family of operators are used:

$$(T_bK_sJ_ah)(x) = a^{-\frac{3}{4}}h(A^{-1}(a)S^{-1}(s)(x - b)),$$

where the matrices $A(a)$ and $S(s)$ are given in the following definition, [9].

Definition 2.1. For $a > 0$ and s in $\mathbb{R}$, let:

$$A(a) = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix} \quad \text{and} \quad S(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

Note that $A^{-1}(a) = A(a^{-1})$ and $S^{-1}(s) = S(-s)$. Also, $A(a_1)A(a_2) = A(a_1a_2)$, where $a_1, a_2 > 0$, and $S(s_1)S(s_2) = S(s_1 + s_2)$, where $s_1, s_2 \in \mathbb{R}$.

Definition 2.2. For h in $L^2(\mathbb{R}^2)$, the dilation, translation, modulation and shear operators are defined respectively by:

$(J_a h)(x) = \frac{1}{\sqrt{detA(a)}} h\left(A^{-1}(a)x\right)$, where $a > 0$ and $x \in \mathbb{R}^2$,

$(T_b h)(x) = h(x - b)$, where $x, b \in \mathbb{R}^2$,

$(E_c h)(x) = e^{2\pi i c\cdot x} h(x)$, where $x, c \in \mathbb{R}^2$,

$(K_s h)(x) = h\left(S^{-1}(s)x\right)$, where $s \in \mathbb{R}$ and $x \in \mathbb{R}^2$. Besides $(K_s^T h)(x) = h\left(S^{-T}(s)x\right)$.

From the previous definitions the next lemma is obtained directly.

Lemma 2.3. *The operators J_a, T_b, E_c, K_s preserve the norm in $L^2(\mathbb{R}^2)$.*

Corollary 2.4. *For h in $L^2(\mathbb{R}^2)$, $a > 0$, $s \in \mathbb{R}$ and $b \in \mathbb{R}^2$, we have:*

$$\|T_b K_s J_a h\|_2 = \|h\|_2.$$

Also, the following results come directly from the definition of the Fourier transform.

Lemma 2.5. *For the operators J_a, T_b, K_s, and for h in $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$:*

$\widehat{J_a h} = J_{\frac{1}{a}} \widehat{h}$, *where* $a > 0$,

$\widehat{T_b h} = E_{-b} \widehat{h}$, *where* $b \in \mathbb{R}^2$,

$\widehat{E_c h} = T_c \widehat{h}$, *where* $c \in \mathbb{R}^2$,

$\widehat{K_s h} = K_{-s}^T \widehat{h}$, *where* $s \in \mathbb{R}$.

In this case, the Fourier transform of h is taken as

$$\widehat{h}(\xi) = \int_{\mathbb{R}^2} e^{-2\pi i \xi\cdot x} h(x)dx.$$

The shearlet transform, as well as the wavelet transform, can be defined from the topological point of view, so that a unitary shearlet group representation can be used to obtain the inversion formula.

Definition 2.6. Let $G = \{(a, s, b)|a > 0, s \in \mathbb{R}, b \in \mathbb{R}^2\}$. For (a_1, s_1, b_1) and (a_2, s_2, b_2) in G, define:

$$\begin{aligned}&(a_1, s_1, b_1) \cdot (a_2, s_2, b_2)\\ &= (a_1a_2, s_1 + s_2\sqrt{a_1}, b_1 + S(s_1)A(a_1)b_2).\end{aligned}$$

Remark 2.7. With this product G becomes a locally compact topological group with identity $(1, 0, 0)$, where $(a, s, b)^{-1} = (\frac{1}{a}, -\frac{s}{\sqrt{a}}, -A^{-1}(a)S^{-1}(s)b)$ is the inverse of (a, s, b). Moreover, the left Haar measure is $d(a, s, b) = \frac{1}{a^3} dadsdb$, and the right Haar measure is $d_r(a, s, b) = \frac{1}{a} dadsdb$ [9]. That is, G is a non-unimodular group.

Remark 2.8. The shearlet group is isomorphic to the locally compact group $G \times \mathbb{R}^2$, where:

$$G = \{S(s)A(a)|a > 0, s \in \mathbb{R}\}.$$

Thus, it is a subgroup of the following group of rotations $GL_2(\mathbb{R}) \times \mathbb{R}^2$ with multiplication defined by $(M, b) \cdot (M', b') = (MM', b + Mb')$.

For (a, s, b) in G the three parameter family of operators is defined as:

$$U(a, s, b) = T_b K_s J_a.$$

In this case, for $h \in L^2(\mathbb{R}^2)$:

$$\begin{aligned}U(a, s, b)h(x) &= (T_b K_s J_a h)(x) = (K_s J_a h)(x - b),\\ &= (J_a h)(S^{-1}(s)(x - b)),\\ &= a^{-\frac{3}{4}} h(A^{-1}(a)S^{-1}(s)(x - b)).\end{aligned}$$

Moreover, U is a unitary representation of G acting on $L^2(\mathbb{R}^2)$.

Definition 2.9. A function h in $L^2(\mathbb{R}^2)$ is admissible if:

$$\int_G |\langle h, U(a, s, b)h\rangle|^2 \, d(a, s, b) < \infty.$$

Lemma 2.10. *Suppose that f, h are in $L^2(\mathbb{R}^2)$, then:*

$$\int_G |\langle f, U(a, s, b)h\rangle|^2 \, d(a, s, b) = C_h \|f\|^2,$$

where

$$C_h := \int_{\mathbb{R}^2} \left|\widehat{h}(k_1, k_2)\right|^2 \frac{1}{k_1^2} dk_1 dk_2.$$

Proof. See [9]. □

Remark 2.11. From Lemma 2.10, we have that h in $L^2(\mathbb{R}^2)$ is admissible if and only if:

$$C_h := \int_{\mathbb{R}^2} \left|\widehat{h}(k_1,k_2)\right|^2 \frac{1}{k_1^2} dk_1 dk_2 < \infty. \quad (2.1)$$

Definition 2.12. Let h be an admissible function h in $L^2(\mathbb{R}^2)$, and let (a,s,b) be in G. The continuous shearlet transform with respect to h is defined as the map:

$$S_h(a,s,b) : L^2(\mathbb{R}^2, dx) \to L^2(G, d(a,s,b)),$$

such that for f in $L^2(\mathbb{R}^2)$:

$$(S_h f)(a,s,b) = \langle f, U(a,s,b)h\rangle = \langle f, T_b K_s J_a h\rangle .$$

That is:

$$(S_h f)(a,s,b) = \int_{\mathbb{R}^2} f(x) \frac{1}{\sqrt{det A(a)}} \overline{h}\left(A^{-1}(a) S^{-1}(s)(x-b)\right) dx.$$

The continuous shearlet transform can be expressed as convolution, as the following remark states.

Remark 2.13. Let h be admissible in $L^2(\mathbb{R}^2)$. Then for $f \in L^2(\mathbb{R}^2)$ and $(a,s,b) \in G$:

$$(S_h f)(a,s,b) = \left[(K_s J_a \overline{h})^{\sim} * f\right](b), \quad (2.2)$$

where the symbol $^{\sim}$ means $h^{\sim}(x) = h(-x)$.

The next result corresponds to the inverse shearlet transform, [9].

Lemma 2.14. *If f, h are in $L^2(\mathbb{R}^2)$, and $(a,s,b) \in G$, then:*

$$f = \frac{1}{C_h} \int_G (S_h f)(a,s,b) U(a,s,b) h \, d(a,s,b),$$

where the convergence is in the weak sense.

Remark 2.15. In the case of band limited shearlets, that is when $supp\,\widehat{h}$ is compact, the function $h \in L^2(\mathbb{R}^2)$ is taken as:

$$\widehat{h}(w) = \widehat{h}(w_1,w_2) = \widehat{h}_1(w_1)\widehat{h}_2\left(\frac{w_2}{w_1}\right),$$

where $w = (w_1,w_2) \in \widehat{\mathbb{R}^2}$, with $w_1 \neq 0$, and where h_1 is a continuous wavelet, $\widehat{h}_1 \in C^\infty(\mathbb{R})$, and $supp\,\widehat{h}_1 \subseteq [-2, -\frac{1}{2}]$ and where h_2 is such that $\widehat{h}_2 \in C^\infty(\mathbb{R})$ and $supp\,\widehat{h}_2 \subseteq [-1,1]$. This generating function was used in [10] to show that the continuous shearlet transform resolves the wave front set.

Moreover, this function satisfies the admissibility condition given in (2.1). That is $h \in L^2(\mathbb{R}^2)$ is admissible if $\widehat{h}(w) = \widehat{h}(w_1,w_2) = \widehat{h}_1(w_1)\widehat{h}_2\left(\frac{w_2}{w_1}\right)$, with $w_1 \neq 0$ and $h_1 \in L^2(\mathbb{R})$ satisfies $\int_0^\infty \left|\widehat{h}_1(a\xi)\right|^2 \frac{da}{a} = 1$, for a.e. $\xi \in \mathbb{R}$, and $\|h_2\|_2 = 1$. [10].

3 Discrete Shearlet Transform

To define the discrete shearlet transform discrete values for the dilation, shear and translation parameters are considered. In this paper this transform is applied to analyze the singularities of functions in $L^2(\mathbb{R}^2)$ by means of the decay of the discrete shearlet transform. For this purpose, similar matrices are considered like the ones given in Definition 2.1 with $a = 4$, and $s = -1$.

Definition 3.1. Consider the following four matrices:

$$A_1 = \begin{pmatrix} \frac{1}{2^2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2^2} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Then for $j, k \in \mathbb{Z}$:

$$A_1^j = \begin{pmatrix} \frac{1}{2^{2j}} & 0 \\ 0 & \frac{1}{2^j} \end{pmatrix}, \quad B_1^k = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix},$$

$$A_2^j = \begin{pmatrix} \frac{1}{2^j} & 0 \\ 0 & \frac{1}{2^{2j}} \end{pmatrix}, \quad B_2^k = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}.$$

Moreover:

$$A_1^{-j} = \begin{pmatrix} 2^{2j} & 0 \\ 0 & 2^j \end{pmatrix}, \quad B_1^{-k} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix},$$

$$A_2^{-j} = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{2j} \end{pmatrix}, \quad B_2^{-k} = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}.$$

Definition 3.2. Consider the group G of the form:

$$G = \{(M, z) : M \in GL_2(\mathbb{R}), z \in \mathbb{R}^2\},$$

where M is the set of matrices of size 2×2 of the form:

$$M = M_d(jk) = A_d^j B_d^k,$$

where $j \in \mathbb{Z}$, $k \in \mathbb{Z}$, and $d = 1, 2$.

The group law in G is given by $(M, z) \cdot (M', z') = (MM', Mz' + z)$, where the inverse is $(M, z)^{-1} = (M^{-1}, -M^{-1}z)$, and the identity is $(I, 0)$.

Remark 3.3. Note that for $d = 1$:

$$M_1(jk) = A_1^j B_1^k = \begin{pmatrix} \frac{1}{2^{2j}} & 0 \\ 0 & \frac{1}{2^j} \end{pmatrix} \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2^{2j}} & -\frac{k}{2^{2j}} \\ 0 & \frac{1}{2^j} \end{pmatrix},$$

and for $d = 2$:

$$M_2(jk) = A_2^j B_2^k = \begin{pmatrix} \frac{1}{2^j} & 0 \\ 0 & \frac{1}{2^{2j}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2^j} & 0 \\ -\frac{k}{2^{2j}} & \frac{1}{2^{2j}} \end{pmatrix}.$$

Hence, note that for $d = 1, 2$:

$$det\, M_d(jk) = det\, A_d^j = \frac{1}{2^{3j}}. \qquad (3.1)$$

Definition 3.4. For h in $L^2(\mathbb{R}^2)$, and each $d = 1, 2$, define the dilation and translation operators respectively by:

$$(J_{M_d(jk)}h)(x) = \frac{1}{\sqrt{det\, M_d(jk)}} h\left(M_d^{-1}(jk)x\right),$$

where $x \in \mathbb{R}^2$ and $j, k \in \mathbb{Z}$.

$(T_{M_d(jk)l}h)(x) = h(x - M_d(jk)l)$, where $x \in \mathbb{R}^2$, $j, k \in \mathbb{Z}$, and $l \in \mathbb{Z}^2$.

Remark 3.5. For $d = 1, 2$, the operators $J_{M_d(jk)}$ and $T_{M_d(jk)l}$ preserve the norm in $L^2(\mathbb{R}^2)$. Moreover, the adjoints are their inverses respectively.

Definition 3.6. $h \in L^2(\mathbb{R}^2)$ is admissible [9], if:

$$0 < C_h := \int_{\mathbb{R}} \int_{\mathbb{R}} \left|\widehat{h}(y_1, y_2)\right|^2 \frac{1}{y_1^2} dy_1\, dy_2 < \infty.$$

Following [11], the discrete shearlet transform is defined as:

Definition 3.7. Let h be an admissible function in $L^2(\mathbb{R}^2)$, and for each $d = 1, 2$, let $(M_d(jk), M_d(jk)l)$ in $GL_2(\mathbb{R}) \times \mathbb{R}^2$. Then the discrete shearlet transform of f in $L^2(\mathbb{R}^2)$, with respect to h is defined as:

$$(\mathcal{D}_h f)(M_1(jk), M_2(jk), M_1(jk)l, M_2(jk)l) = \sum_{d=1}^{2} \left\langle f, T_{M_d(jk)l} J_{M_d(jk)} h \right\rangle.$$

Definition 3.8. The family of functions:

$$\left\{T_{M_d(jk)l} J_{M_d(jk)} h\right\},$$

in $L^2(\mathbb{R}^2)$ with $j, k \in \mathbb{Z}, l \in \mathbb{Z}^2$ and where $d = 1, 2, j \geq 0$ is a tight frame [11], if there is a positive constant C such that for any $f \in L^2(\mathbb{R}^2)$:

$$C\|f\|_2^2 = \sum_{j,k,l,d} \left|\left\langle f, T_{M_d(jk)l} J_{M_d(jk)} h \right\rangle\right|^2.$$

Theorem 3.9. *For any f in $L^2(\mathbb{R}^2)$ and for a given admissible function $h \in L^2(\mathbb{R}^2)$, if $T_{M_d(jk)l} J_{M_d(jk)} h$ is a tight frame, then there is a constant $C > 0$ such that:*

$$f = \frac{1}{C} \sum_{j,k,l,d} \left\langle f, T_{M_d(jk)l} J_{M_d(jk)} h \right\rangle T_{M_d(jk)l} J_{M_d(jk)} h, \qquad (3.2)$$

where the convergence is in the weak sense, and where $j, k \in \mathbb{Z}, l \in \mathbb{Z}^2$, and $d = 1, 2$. [11].

Remark 3.10. According to Definitions 3.4 and 3.7, the discrete shearlet transform can be written as:

$$\begin{aligned} &(\mathcal{D}_h f)(M_1(jk), M_2(jk), M_1(jk)l, M_2(jk)l) \\ &= \sum_{d=1}^{2} \int_{\mathbb{R}^2} f(x)(T_{M_d(jk)l} J_{M_d(jk)} \overline{h})(x)dx, \\ &= \sum_{d=1}^{2} \int_{\mathbb{R}^2} f(x)(J_{M_d(jk)} \overline{h})(x - M_d(jk)l)dx, \\ &= \sum_{d=1}^{2} \int_{\mathbb{R}^2} f(x) \frac{1}{\sqrt{det\, M_d(jk)}} \overline{h}(M_d^{-1}(jk)x - l)dx. \end{aligned} \qquad (3.3)$$

That is:

$$(\mathcal{D}_h f)(M_1(jk), M_2(jk), M_1(jk)l, M_2(jk)l)$$
$$= \sum_{d=1}^{2} \int_{\mathbb{R}^2} f(x) \frac{1}{\sqrt{det A_d^j}} \overline{h}\left(B_d^{-k} A_d^{-j} x - l\right) dx,$$
$$= \int_{\mathbb{R}^2} f(x) \frac{1}{\sqrt{det A_1^j}} \overline{h}\left(B_1^{-k} A_1^{-j} x - l\right) dx,$$
$$+ \int_{\mathbb{R}^2} f(x) \frac{1}{\sqrt{det A_2^j}} \overline{h}\left(B_2^{-k} A_2^{-j} x - l\right) dx,$$
$$= \int_{\mathbb{R}^2} \frac{f(x)}{2^{\frac{3}{2}j}} \overline{h}\left(2^{2j} x_1 + k 2^j x_2 - l_1, 2^j x_2 - l_2\right) dx,$$
$$+ \int_{\mathbb{R}^2} \frac{f(x)}{2^{\frac{3}{2}j}} \overline{h}\left(2^j x_1 - l_1, 2^{2j} x_2 + k 2^j x_1 - l_2\right) dx.$$

Lemma 3.11. *The discrete shearlet transform can be expressed as a convolution. That is:*

$$(\mathcal{D}_h f)(M_1(jk), M_2(jk), M_1(jk)l, M_2(jk)l)$$
$$= \sum_{d=1}^{2} \left[\left(J_{M_d(jk)} \overline{h}\right)^{\sim} * f \right] (M_d(jk)l),$$

where $\sim$ *means* $\phi^{\sim}(x) = \phi(-x)$.

Proof. From (3.3):

$$\sum_{d=1}^{2} \left[\left(J_{M_d(jk)} h\right)^{\sim} * f \right] (M_d(jk)l),$$
$$= \sum_{d=1}^{2} \int_{\mathbb{R}^2} \left(J_{M_d(jk)} \overline{h}\right)^{\sim} (M_d(jk)l - x) f(x) dx,$$
$$= \sum_{d=1}^{2} \int_{\mathbb{R}^2} \left(J_{M_d(jk)} \overline{h}\right) (x - M_d(jk)l) f(x) dx,$$
$$= \sum_{d=1}^{2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{det A_d^j}} \overline{h}(M_d(jk)^{-1} x - l) f(x) dx,$$
$$= (\mathcal{D}_h f)(M_1(jk), M_2(jk), M_1(jk)l, M_2(jk)l).$$

□

4 Partial Result

Lemma 4.1. *Suppose* h *be in* $C_0(\mathbb{R}^2)$ *is an admissible function not identically zero, such that* $\int_{\mathbb{R}} h(x) dx = 0$. *Consider* f *in* $L^2(\mathbb{R}^2)$, *and for each* $d = 1, 2$, *let:*

$$(\mathcal{P}_h^{(d)} f)(M_d(jk), M_d(jk)l)$$
$$:= \frac{1}{\sqrt{det\, M_d(jk)}} \left\langle f, T_{M_d(jk)l} J_{M_d(jk)} h \right\rangle.$$

If f *is continuous in a neighborhood of* $x = 0 \in \mathbb{R}^2$, *then for each* $d = 1, 2$, *and any* $j, k \in Z$ *and any* $l \in \mathbb{Z}^2$*:*

$$\lim_{(M_d(jk), M_d(jk)l) \to (0,0)} (\mathcal{P}_h^{(d)} f)(M_d(jk), M_d(jk)l) = 0.$$

Proof. Note that if $j \nrightarrow +\infty$, then from (3.1), we have $det M_d(jk) \nrightarrow 0$. Hence, from (2.2), and for any $d = 1, 2$, the function $\mathcal{P}_h^{(d)} f$ is continuous for any $(M_d(jk), M_d(jk)l) \in GL_2(\mathbb{R}) \times \mathbb{R}^2$.

Consider now the case when $j \to +\infty$, then $det M_d(jk) \to 0$. Thus, by hypothesis suppose that f is continuous in a neighborhood of $x = 0$ containing the closed ball $\overline{B_R(0)}$, where $R > 0$, and take $M_d(jk)l$ in the open ball $B_{\frac{R}{2}}(0)$.

Now, since $h \in C_0(\mathbb{R}^2)$ there is $L > 0$ such that $supp\, h \subset B_L(0)$. Then since the adjoints of the operators $T_{M_d(jk)l}, J_{M_d(jk)}$ are their inverses respectively (Remark 3.5), then for each $d = 1, 2$:

$$(\mathcal{P}_h^{(d)} f)(M_d(jk), M_d(jk)l)$$
$$= \frac{1}{\sqrt{det M_d(jk)}} \left\langle f, T_{M_d(jk)l} J_{M_d(jk)} h \right\rangle,$$
$$= \frac{1}{\sqrt{det M_d(jk)}} \left\langle J_{M_d(jk)}^{-1} T_{M_d(jk)l}^{-1} f, h \right\rangle,$$
$$= \frac{1}{\sqrt{det M_d(jk)}} \int_{B_L(0)} (J_{M_d(jk)}^{-1} T_{M_d(jk)l}^{-1} f)(x) \overline{h}(x) dx,$$
$$= \int_{B_L(0)} T_{M_d(jk)l}^{-1} f(M_d(jk)x) \overline{h}(x) dx,$$
$$= \int_{B_L(0)} f(M_d(jk)x + M_d(jk)l) \overline{h}(x) dx.$$

Since $\int_{\mathbb{R}} h(x)dx = 0$, and f is continuous near 0, it follows that for any $j, k \in \mathbb{Z}$ and any $l \in \mathbb{Z}^2$:

$$\lim_{(M_d(jk), M_d(jk)l) \to (0,0)} (\mathcal{P}_h^{(d)} f)(M_d(jk), M_d(jk)l),$$
$$= f(0) \int_{B_L(0)} \overline{h}(x)dx = f(0) \cdot 0 = 0.$$

□

5 Main Result

Theorem 5.1. *Suppose h be in $C_0(\mathbb{R}^2)$ is an admissible function not identically zero, such that $\int_{\mathbb{R}} h(x)dx = 0$. For $d = 1,2$ suppose that $T_{M_d(jk)l} J_{M_d(jk)} h$ is a tight frame. Consider f in $L^2(\mathbb{R}^2)$ and $(M_d(jk), z) \in GL_2(\mathbb{R}) \times \mathbb{R}^2$.*

If for each $d = 1,2$:

$$\lim_{(M_d(jk),\, z_d) \to (0,\, z'_d)} (\mathcal{P}_h^{(d)} f)(M_d(jk), z_d),$$

exists for each k in $[-Q, Q]$ for some positive $Q \in \mathbb{Z}$ and any z'_d in an open neighborhood of $x = 0 \in \mathbb{R}^2$, then f in $L^2(\mathbb{R}^2)$ is continuous in a neighborhood of $x = 0 \in \mathbb{R}^2$.

Proof of Theorem 5.1. Suppose that for each $d = 1,2$:

$$\lim_{(M_d(jk),\, z_d) \to (0,\, z'_d)} (\mathcal{P}_h^{(d)} f)(M_d(jk), z) := F_d(0,\, z'_d), \tag{5.1}$$

exists for each k in $[-Q, Q]$ and any z'_d in an open neighborhood containing the closed ball $\overline{B_R(0)}$, with $R > 0$.

Now for fixed x in the open ball $B_R(0)$ and $y \in \mathbb{R}^2$, for each $d = 1,2$ let:

$$\mathcal{I}_d(M_d(jk), x, y),$$
$$= \begin{cases} h(-y)(\mathcal{P}_h^{(d)} f)(M_d(jk), x + M_d(jk)y), \\ \text{if} \quad j \nrightarrow -\infty, \quad \text{and} \\ h(-y) F_d(M_d(jk), x) \quad \text{if} \quad j \to -\infty. \end{cases} \tag{5.2}$$

Note that for such x, the function $\mathcal{I}_d$ is well-defined for all $j \in \mathbb{Z}, k \in [-Q, Q]$, and $y \in \mathbb{R}^2$.

Then we have the following three Claims.

Claim 1. For each $d = 1,2$, the function $\mathcal{I}_d$ is continuous in $GL_2(\mathbb{R}) \times \overline{B_R(0)} \times \mathbb{R}^2$.

Proof. See Appendix □

Claim 2. For each $d = 1,2$ and fixed $x \in \overline{B_R(0)}$, the triple series:

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^2} \mathcal{I}_d \left(M_d(jk), x, l - M_d^{-1}(jk)x \right),$$

converges uniformly on $\overline{B_R(0)}$.

Proof. See Appendix □

Claim 3. For each $d = 1,2$ and $x \in \overline{B_R(0)}$, the function:

$$W_d(x) := \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^2} \mathcal{I}_d \left(M_d(jk), x, l - M_d^{-1}(jk)x \right),$$

is continuous at $x = 0$.

Proof. See Appendix □

Back to the proof of Theorem 5.1, for any integer $r \geq 0$, any $x \in \mathbb{R}^2$, and for each $d = 1,2$, define:

$$U_{d,r}(x) := \sum_{j=-r}^{r} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^2} \langle f, T_{M_d(jk)l} J_{M_d(jk)} h \rangle \cdot \frac{1}{\sqrt{det M_d(jk)}} h \left(M_d(jk)x - l \right).$$

Then by Claim 3, for each $d = 1,2$ and for any: $x \in \overline{B_R(0)}$,

$$\lim_{r \to \infty} U_{d,r}(x) = W_d(x).$$

That is, for each $d = 1,2$, it follows that $U_{d,r} \to W_d$ pointwise on $B_R(0)$ as $r \to \infty$. Hence, $U_{1,r} + U_{2,r} \to W_1 + W_2$ pointwise on $B_R(0)$ as $r \to \infty$.

On the other hand from (3.2), $U_{1,r} + U_{2,r} \to Cf$ weakly in $L^2(M_{22} \times \mathbb{R}^2)$. Hence, $f = \frac{1}{C}(W_1 + W_2)$ almost everywhere, and due to Claim 3, since W_1 and W_2 are continuous at $x = 0$, then f is continuous at $x = 0$.

This completes the proof of Theorem 5.1. □

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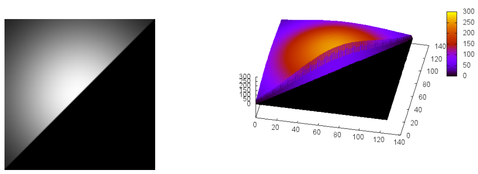

Fig. 1. Image IMG1 with discontinuity along $x_1 = x_2$. Top view projection as gray scale heat map (left) and a colored angular proyection (right)

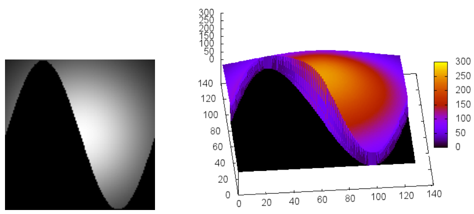

Fig. 2. Image IMG2 with discontinuity along $x_1 < 64(1 + \sin(\pi x_2/64))$. Top view projection as grayscale heat map (left) and a colored angular proyection (right)

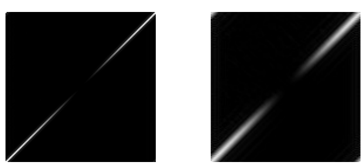

Fig. 3. Shearlet coefficients illustration for IMG1: black pixels for continuous sections and clear pixels along discontinuities

6 Experiments

To illustrate the main results given in Lemma 4.1 and Theorem 5.1, grayscale images with width W and height H were processed. Grayscale pixel values are in $[0, 255]$ where 0 means a black pixel and 255 means a white pixel. Note that, although $f \in L^2(\mathbb{R}^2)$, pixel values are integer and the energy of the image is given by the sum of the square values of the pixels.

An image IMG with $W = H = 128$ was built from:

$$f(x_1, x_2) = 255 * e^{-\frac{(x_1-64)^2+(x_2-64)^2}{512}},$$

by getting integer values. Since IMG represents a continuous gaussian function, a cross section was produced with zero values for $x_1 < x_2$ to get image IMG1 with two continuous sections, as is shown in Figure 1.

In a similar way, image IMG2 was built from IMG with zero values for $x_1 < 64(1 + \sin(\pi x_2/64))$. Figure 2 shows a top view projection for IMG2 (left) as grayscale heat map and a colored angular projection (right).

In both cases, for IMG1 and IMG2, the insertion of zero values aims to define black regions and discontinuities to be studied. Shearlab [12] software was used to get the shearlet coefficients for IMG1 and IMG2. For a single shearlet scale, $N = 9$ decomposition bands are obtained for each image. To illustrate how the shearlet transform coefficients tends to zero in continuous sections, shearlet coefficients were scaled to $[0, 255]$ to appreciate a 128×128 grayscale heat map where dark pixels mean close to zero values (shearlet transform converges) and discontinuities (where shearlet transform does not converge) are perceived as non-black pixels ("clear" pixels that tends to white).

Note that by translation to any point $x \in \mathbb{R}^2$, and not only at $(0, 0)$, the results described in Lemma 4.1 and Theorem 5.1 show that:

1. If we take a point in the black zone of the shearlet transform (where it tends/converges to zero) then by Theorem 5.1 the function is continuous.

2. If we take a point in the white region of the shearlet transform (the shearlet coefficients takes large values, meaning that it does not converge) then by Lemma 4.1 the function is not continuous.

For IMG1, two illustrative shearlet images were chosen as high-frequency bands and they are shown in Figure 3 where clear pixels follow the line corresponding to $x_1 = x_2$ as it was expected from Figure 1.

For IMG2, eight shearlet images were chosen (non-low pass frequency bands) and they are shown in Figure 4 where there are clear pixels along $x_1 = 64(1 + \sin(\pi x_2/64))$ by detecting the discontinuity.

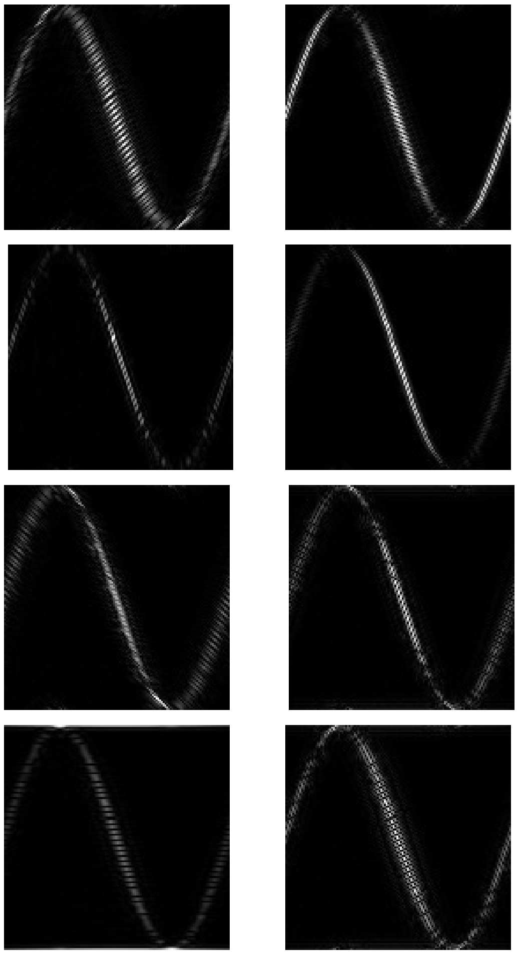

Fig. 4. Shearlet illustration with black pixels on continuous sections and clear pixels along a sinusoidal discontinuity. Each subfigure refers to a different direction

Note that subfigures of Figure 4 have distinct directionality of shearlets and non-dark pixels line up to 8 different directions.

To expose the directionality of shearlets an image IMG3 with zero values except at $(63, 63)$ with a 255 value was generated to simulate a pulse embedded in 128×128 pixels. The corresponding images built from the shearlet coefficients are shown in Figure 5. Note the directionality change counterclockwise when reading subfigures from left to right and top to bottom in Figure 5.

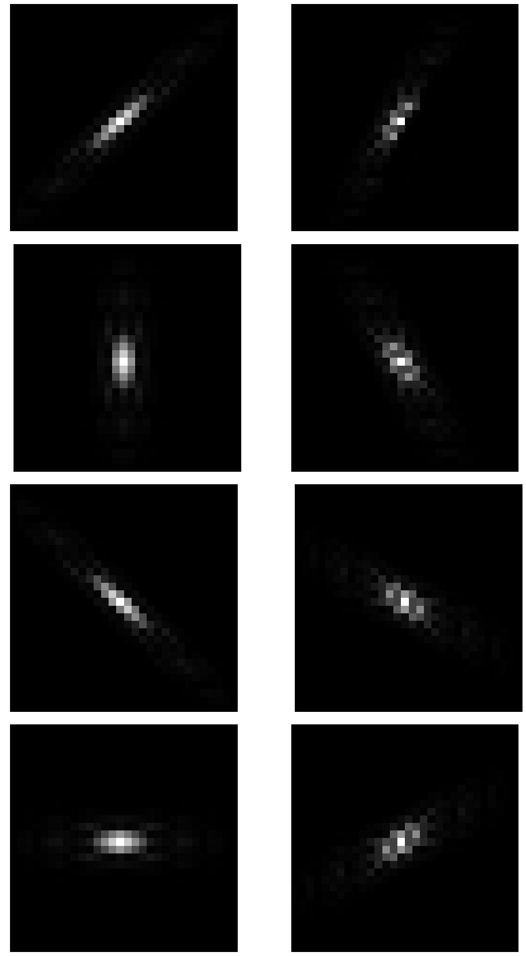

Fig. 5. Illustration of the shearlet directionality for IMG3 from a centered pulse. Zoom to 30×30 pixels

Additionally, the 2D discrete shearlet transform was applied to the "Barbara" image (see Figure 6) and from the shearlet coefficients 8 subfigures were generated (see Figure 7) where it is possible to appreciate pixel patterns that match the different directions illustrated in Figure 5.

7 Conclusions

There are several works about application of shearlets, in particular in two dimensions for images with edges.

Fig. 6. Barbara grayscale image with 128×128 pixels

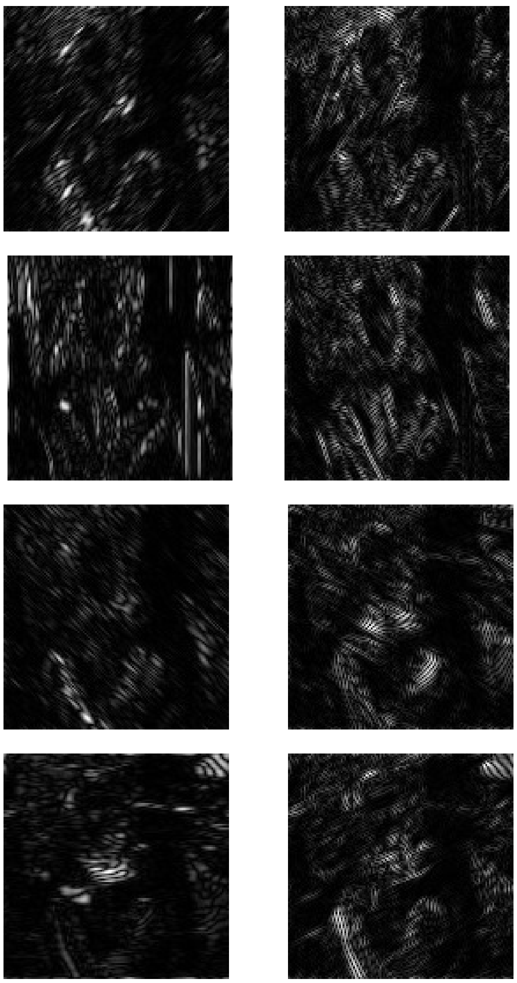

Fig. 7. Illustration of the shearlet directionality for Barbara

This manuscript aims to support these applications, once it was shown that it is possible to study the continuity of a function in two dimensions through the convergence of the 2D discrete shearlet transform. Close to zero shearlet coefficients are associated to continuous sections in images, whereas high values of shearlet coefficients reveal the edges.

Both properties of detecting discontinuities and providing directionality inside images, make the discrete shearlet transform an interesting and useful tool in applications such as image classification and opens the possibility to extend these results to higher dimensions.

8 Appendix A

Proof of Claim 1. If $j \nrightarrow -\infty$, then from (5.2):

$$
\begin{aligned}
&\mathcal{I}_d(M_d(jk), x, y),\\
&= h(-y)(\mathcal{P}_h^{(d)} f)(M_d(jk), x + M_d(jk)y),\\
&= h(-y)\frac{1}{\sqrt{detM_d(jk)}} \left\langle f, T_{x+M_d(jk)y} J_{M_d(jk)} h \right\rangle,\\
&= h(-y)\frac{1}{\sqrt{detM_d(jk)}},\\
&\qquad \cdot \left[\left(J_{M_d(jk)} h\right)^{\sim} * f \right](x + M_d(jk)y).
\end{aligned}
$$

Since $h \in C_0(\mathbb{R}^2)$ and $f \in L^2(\mathbb{R}^2)$, then the convolution $\left(J_{M_d(jk)} h\right)^{\sim} * f$ is a continuous function. Then for each $d = 1, 2$, the function $\mathcal{I}_d$ is continuous in $GL(\mathbb{R}) \times \overline{B_R(0)} \times \mathbb{R}^2$.

In this case, the Frobenious norm is taken in $GL(\mathbb{R})$, where $\|A\| = \langle A, A \rangle^{\frac{1}{2}}$.

Now if $j \to -\infty$, then from (5.1), for any $(0, x_1, y_1) \in GL(\mathbb{R}) \times \overline{B_R(0)} \times \mathbb{R}^2$:

$$
\begin{aligned}
&\lim_{(M_d(jk),x,y) \to (0,\, x_1, y_1)} \mathcal{I}_d(M_d(jk), x, y),\\
&= \lim_{(M_d(jk),x,y) \to (0,\, x_1, y_1)} h(-y),\\
&\qquad \cdot (\mathcal{P}_h^{(d)} f)(M_d(jk), x + M_d(jk)y),\\
&= h(-y_1) \lim_{(M_d(jk),z_d) \to (0,\, x_1)} (\mathcal{P}_h^{(d)} f)(M_d(jk), z_d),\\
&= h(-y_1) F_d(0, x_1) = \mathcal{I}_d(0, x_1, y_1).
\end{aligned}
$$

Therefore, for each $d = 1, 2$, the function $\mathcal{I}_d$ is continuous in $GL(\mathbb{R}) \times \overline{B_R(0)} \times \mathbb{R}^2$. □

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Proof of Claim 2. Note that if $j \nrightarrow \infty$, and if for each $d = 1, 2$, take $-y = -l + M_d^{-1}(jk)x$, then from (5.2):

$$\begin{aligned}&\mathcal{I}_d(M_d(jk), x, l - M_d^{-1}(jk)x),\\&= h(-l + M_d^{-1}(jk)x)\frac{1}{\sqrt{detM_d(jk)}},\\&\quad\cdot \left\langle f, T_{M_d(jk)l} J_{M_d(jk)} h \right\rangle.\end{aligned}$$

Then, from (3.1):

$$\begin{aligned}&\left|\mathcal{I}_d(M_d(jk), x, l - M_d^{-1}(jk)x)\right|,\\&\leq \frac{\|f\|_2 \ \|h\|_2}{2^{\frac{3}{2}j}} \left|h(-l + M_d^{-1}(jk)x)\right|.\end{aligned}$$

Now for a given positive integer V, define:

$$G_d(M_d(jk), l - M_d^{-1}(jk)x), \\ := \begin{cases} \left|\mathcal{I}_d(M_d(jk), x, l - M_d^{-1}(jk)x)\right|, \\ \qquad \text{if } j \in [-V, V], \\ \frac{\|f\|_2 \ \|h\|_2}{2^{\frac{3}{2}j}} \ \left|h(-l + M_d^{-1}(jk)x)\right|, \\ \qquad \text{if } j \notin [-V, V]. \end{cases} \tag{8.1}$$

Hence:

$$\begin{aligned}&\left|\mathcal{I}_d(M_d(jk), x, l - M_d^{-1}(jk)x)\right|,\\&\leq G_d(M_d(jk), l - M_d^{-1}(jk)x),\end{aligned}$$

for all $(M_d(jk), l - M_d^{-1}(jk)x) \in GL_2(\mathbb{R}) \times \mathbb{R}^2$.

On the other hand, since $h \in C_0(\mathbb{R}^2)$ there is $L > 0$ such that $supp\, h \subset \overline{B_L(0)}$. So, there is a positive integer $N > L$ so that $h(-l + M_d^{-1}(jk)x) = 0$ for $l = (l_1, l_2) \in \mathbb{Z}^2$ with $l_i > N$ for $i = 1, 2$. Hence, it can be considered the series over $l_i \in \mathbb{Z}$ only from $-N$ to N for $i = 1, 2$, and since $k \in [-Q, Q]$, then:

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^2} \left|G_d(M_d(jk), l - M_d^{-1}(jk)x)\right|,\\&= \left(\sum_{j=-\infty}^{-V-1} + \sum_{j=-V}^{V} + \sum_{j=V+1}^{\infty}\right) \sum_{k=-Q}^{Q} \left(\sum_{l_1=-N}^{N}\sum_{l_2=-N}^{N}\right),\\&\left|G_d(M_d(jk), l - M_d^{-1}(jk)x)\right|.\end{aligned}$$

Due to the fact that $j \nrightarrow -\infty$, then from (8.1):

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^2} \left|G_d(M_d(jk), l - M_d^{-1}(jk)x)\right|,\\&= \sum_{j=-V}^{V}\sum_{k=-Q}^{Q}\left(\sum_{l_1=-N}^{N}\sum_{l_2=-N}^{N}\right),\\&\left|\mathcal{I}_d(M_d(jk), x, l - M_d^{-1}(jk)x)\right|\\&+ \sum_{j=V+1}^{\infty}\sum_{k=-Q}^{Q}\left(\sum_{l_1=-N}^{N}\sum_{l_2=-N}^{N}\right),\\&2^{-\frac{3}{2}j}\|f\|_2 \ \|h\|_2 \ \left|h(-l + M_d^{-1}(jk)x)\right|.\end{aligned} \tag{8.2}$$

Note that since from Claim 1, the function $\mathcal{I}_d$ is continuous, it follows that the series in the first term of (8.2) converges.

On the other hand, if:

$S = Sup\ \left|h(-l + M_d^{-1}(jk)x)\right|$, it follows that the series in the second term of (8.2) converges to:

$$S\|f\|_2 \ \|h\|_2\,(2N+1)^2(2Q+1)\left(\sum_{j=V+1}^{\infty} 2^{-\frac{3}{2}j}\right).$$

Thus, for $d = 1, 2$:

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^2} \left|G_d(M_d(jk), l - M_d^{-1}(jk)x)\right|,$$

converges.

Hence, for fixed $x \in \overline{B_R(0)}$, the triple series:

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^2} \mathcal{I}_d(M_d(jk), x, l - M_d^{-1}(jk)x),$$

converges uniformly. □

Proof of Claim 3. By Claim 1, the function $\mathcal{I}_d$ is continuous on $GL_2(\mathbb{R}) \times \overline{B_R(0)} \times \mathbb{R}^2$, and by Claim 2, the triple series:

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^2} \mathcal{I}_d(M_d(jk), x, l - M_d^{-1}(jk)x),$$

converges uniformly on $\overline{B_R(0)}$.

Then in particular for $x = 0$, the triple series:

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^2}\mathcal{I}_d(M_d(jk),0,l),$$

converges absolutely and uniformly on $\overline{B_R(0)}$.

Hence, for each $d = 1, 2$:

$$\lim_{x\to 0} W_d(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^2}\Big(\lim_{x\to 0}\mathcal{I}_d(M_d(jk),x,l-M_d^{-1}(jk)x)\Big), = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^2}\mathcal{I}_d(M_d(jk),0,l) = W_d(0).$$

This proves Claim 3. □

References

1. **Easley, G., Labate, D., & Lim, W.Q. (2008).** Sparse directional image representations using the discrete shearlet transform. *Appl. Comput. Harmon. Anal.,* Vol. 25, No. 1, pp. 25–46. DOI: 10.1016/j.acha.2007.09.003.
2. **Labate, D., Lim, W.Q., Kutyniok, G., & Weiss, G. (2005).** Sparse multidimensional representation using shearlets. *Proc. SPIE Wavelets XI*, Vol. 5914, pp. 254–262. DOI:10.1117/12.613494.
3. **Guo, K., Kutyniok, G., & Labate, D. (2006).** Sparse multidimensional representations using anisotropic dilation and shear operators. *Wavelets and Splines* **Chen, G., Lai, M., Eds.**, Nashboro Press: Brentwood.
4. **Navarro, J. (1998).** Wavelet transforms and singularities of L_2-functions in $\mathbb{R}^n$. *Revista Colombiana de Matematicas*, Vol. 32, pp. 93–99.
5. **Navarro, J. & Herrera, O. (2012).** Convergence of the discrete wavelet transform. *Int. J. Wavelets Multi.,* Vol. 10, No. 6, pp. 1–12.
6. **Daubechies, I. (1992).** *Ten lectures on wavelets*. SIAM: Philadelphia.
7. **Navarro, J. & Herrera, O. (2016).** Decay properties of the discrete wavelet transform in n dimensions with independent dilation parameters. *J. Inequal. Appl.,* pp. 1–19.
8. **Kugarajah, T. & Qinghua, Z. (1995).** Multidimensional wavelet frames. *IEEE Trans. Neural Netw.,* Vol. 6, No. 6, pp. 1552–1556.
9. **Dahlke, S., Kutyniok, G., Maass, P., Sagiv, C., Stark, H.G., & Teschke, G. (2008).** The uncertainty principle associated with the continuous shearlet transform. *Int. J. Wavelets Multi.,* Vol. 6, No. 2, pp. 157–181.
10. **Kutyniok, G. & Labate, D. (2009).** Resolution of the wavefront set using continuous shearlets. *Trans. Amer. Math. Soc.,* Vol. 36, No. 5, pp. 2719–2754.
11. **Goossens, B., Aelterman, J., Luong, H., Pizurica, A., & Philips, W. (2009).** Efficient design of a low redundant discrete shearlet transform. *LNLA:2009 Int. Workshop on Local and Non-Local approximation in Image Proccesing*, pp. 112–124.
12. **Kutyniok, G., Lim, W.Q. & Reisenhofer, R. (2016).** ShearLab 3D: Faithful digital shearlet transforms based on compactly supported shearlets. *ACM Trans. Math. Software,* Vol. 42, No. 5.

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Corresponding author is Oscar Herrera Alcántara.